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THE MULTIPLICITY FUNCTION OF A LOCAL RING

BY

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ABSTRACT. Let A be a local ring with maximal ideal m. Let $f \in A$, and define $\mu_A(f)$ to be the multiplicity of the A-module A/Af with respect to m. Under suitable conditions $\mu_A(fg) = \mu_A(f) + \mu_A(g)$. The relationship of μ_A to reduction of A, normalization of A and a quadratic transform of A is studied. It is then shown that there are positive integers $n_1, \ldots, n_s$ and rank one discrete valuations $v_1, \ldots, v_s$ of A centered at m such that $\mu_A(f) = n_1 v_1(f) + \cdots + n_s v_s(f)$ for all regular elements f of A.

Let A be a nonnull noetherian local ring with maximal ideal m. Let d be the *(Krull) dimension* of A, the maximal length of a chain of prime ideals of A, excluding A. Let k be the residue field A/m, and let $G_m A$ be the associated graded ring of A with respect to m.

Let $f \in A$. If A/Af is of dimension $d - 1$ define $\mu_A(f)$ to be $e_m(A/Af)$, the multiplicity of the A-module A/Af relative to m in dimension $d - 1$ [6, *p.* V-2] *or the multiplicity of the local ring A/Af* ([7, *p.* 294], *or*[3, *p.* 75]). *If A/Af is of dimension d, define $\mu_A(f)$ to be ∞. Call $\mu_A(f)$ the multiplicity of f (at m in A).*

If A is a regular local ring, μ_A is known to be the order valuation of A [3, 40.2, p. 154]. If A is entire $\mu_A(fg) = \mu_A(f) + \mu_A(g)$ (Proposition 1, §1). The order function v_A of A [7, p. 249] satisfies $v_A(f + g) \geq \min\{v_A(f), v_A(g)\}$, and (Proposition 2, §1) v_A is a valuation if and only if μ_A is a multiple of v_A.

If the ideal (0) is unmixed in A, μ_A is found to extend to the components of A (Lemma 2, §2). If A is of dimension one, μ_A is found to extend to the normalization of A (Lemma 3, §2). The extension of A to the first neighborhood ring of A (a quadratic transform of A) is found to preserve μ_A (Lemma 4, §3).

This is used to prove the theorem of §4, that there are positive integers $n_1, \ldots, n_s$ and discrete rank one valuations $v_1, \ldots, v_s$ of A centered at m such that for every regular element f of A

$$\mu_A(f) = n_1 v_1(f) + \cdots + n_s v_s(f).$$

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The valuations $v_1, \ldots, v_s$ arise from (dimension one) normalization of the first neighborhood ring of A, and each n_i is the product of the length of a primary component of (0) in A of dimension d, the multiplicity of a d-dimensional component of the tangent cone of A at the origin, the index of a normalization and another factor arising from a nonfinite normalization of an entire local ring of dimension one.

Let p be a prime ideal of the noetherian ring A. The *depth* of p will denote throughout the Krull dimension of A/p.

1. **Elementary properties of μ_A.** For an A-module M let $l_A(M)$ denote the length of M as an A-module. If p is a prime ideal of A and if $\mathfrak{A}$ is an ideal of A let $\lambda_p(\mathfrak{A}) = l_{A_p}(A_p/A_p\mathfrak{A})$.

PROPOSITION 1. *Let f and g be two elements of a local ring A, and assume either that f is a regular element of A or that $\mu_A(f) = \infty$. Then*

$$\mu_A(fg) = \mu_A(f) + \mu_A(g).$$

PROOF. If $\mu_A(f) = \infty$, then f and fg are contained in a prime ideal of A of depth d, and $\mu_A(fg) = \infty$.

Let f be a regular element of A and assume that $\mu_A(g)$ is finite. By [6, p. V-3], for any $h \in A$ such that $\mu_A(h)$ is finite,

$$\mu_A(h) = \sum_p \lambda_p(Ah)\, e_m(A/p)$$

where the sum ranges over all prime ideals p of A of depth $d - 1 = \dim A - 1$,

$$0 \longrightarrow Af/Afg \longrightarrow A/Afg \longrightarrow A/Af \longrightarrow 0$$

is exact, $Af/Afg \simeq A/Ag$ as A-modules, $\lambda_p(Afg) = \lambda_p(Af) + \lambda_p(Ag)$, and the proposition follows.

REMARK. Let $A = k[x, y]_{(x,y)} = k[X, Y]_{(X,Y)}/(X^2, XY)$. By direct computation $\mu_A(y) = 3$ and $\mu_A(y^2) = 5$. Thus $\mu_A(fg)$ need not be $\mu_A(f) + \mu_A(g)$ if neither f nor g is regular and if both $\mu_A(f)$ and $\mu_A(g)$ are finite.

PROPOSITION 2. *Let A be an entire local ring and suppose the order function v_A of A is a valuation. Then*

$$\mu_A = e_m(A)\, v_A.$$

PROOF. G_mA is entire, and if f is a nonzero element of A, f is superficial of degree $v_A(f)$. Thus [7, Lemma 4, p. 286], $\mu_A(f) = e_m(A/Af) = e_m(A) \cdot v_A(f)$.

COROLLARY. *If A is a regular local ring then μ_A is the order valuation.*

REMARK. Let A be an entire local ring of dimension one and suppose the order function v_A of A is a valuation. Then $G_m A$ is an entire graded ring over $k = A/m$ of dimension one which must be the polynomial ring in one variable over k, $\dim_k m/m^2 = 1$, A is therefore a regular local ring, and $\mu_A = v_A$.

The following proposition gives a geometric definition of μ_A. The local ring A is said to be *affine* if it is the homomorphic image of a localization of a polynomial ring over a field.

PROPOSITION 3. *Let A be an entire affine local ring which has an infinite residue field $k = A/m$. Then A is the homomorphic image of an affine regular local ring B. Let p be the kernel of this homomorphism of B onto A, which is local, and notice that B is equicharacteristic with residue field k. Let d be the dimension of A. Then for every regular element f of A,*

$$\mu_A(f) = \min_{f_1,\dots,f_{d-1}} \{i(Z(B/p) \cdot Z(B/Bf_1) \cdots Z(B/Bf_{d-1}) \cdot Z(B/Bf), m)\}$$

where the minimum is taken over all $f_1, \dots, f_{d-1} \in A$ for which the intersection is proper. For the definition and notation of the right-hand side of the equation see [1] *and* [6, §V–C].

REMARK. By applying Lemma 2, §2 to $\mu_A(f) = e_{(f,f_1,\dots,f_{d-1})}(A)$, by the additivity of $Z(B/p)$ and the linearity of $i(\cdot, m)$, the hypothesis that A be entire may be dropped from Proposition 3.

REMARK. This proposition does not necessarily hold if the residue field is finite. For let k be the field of p^n elements, and let $A = k[X_1, X_2]_{(X_1,X_2)}$. Letting μ' denote the formula of the right-hand side of the equality of the proposition, $\mu'(X_2(\Pi_{\alpha\in k}(X_1 - \alpha X_2))) = p^n + 2$, whereas $\mu_A(X_2(\Pi_{\alpha\in k}(X_1 - \alpha X_2))) = p^n + 1$.

PROOF OF PROPOSITION 3.

$$\mu_A(f) = e_{(f_1,\dots,f_{d-1})}(A/Af)$$

for some $f_1, \dots, f_{d-1} \in m$ [7, Theorem 22, p. 294]

$$= \min_{f_1,\dots,f_{d-1}} \{e_{(f_1,\dots,f_{d-1})}(A/Af)\}$$

where $(f_1, \dots, f_{d-1})$ is an open ideal of A/Af [7, Lemma 2, p. 285]. The elements $f_1, \dots, f_{d-1}$ have representatives in B and in A, and consider $f_1, \dots, f_{d-1}$ to be in either B, A or A/Af.

Let M be the maximal ideal of B, let $\hat{B}$ be the M-adic completion of B, and let $\hat{p} = \hat{B}p$. $\hat{A} = \hat{B}/\hat{p}$. $\hat{B} \simeq k[[X_1, \dots, X_n]]$ for some n. Let $(f_1, \dots, f_{d-1})$ be an open ideal of A/Af.

$$e_{(f_1,\dots,f_{d-1})}(A/Af) = e_{(f_1,\dots,f_{d-1},f)}(A)$$

([4, p. 300] for $((0) :_A Af) = (0)$)

$$= e_{(f_1,\dots,f_{d-1},f)}(\hat{B}/\hat{p})$$

$$= e_{(f_1 \otimes 1,\dots,f_{d-1}\otimes 1, f\otimes 1)}$$

$$((\hat{B} \hat{\otimes}_k \hat{B}/\hat{p})/(X_1 \otimes 1 - 1 \otimes X_1, \dots, X_n \otimes 1 - 1 \otimes X_n))$$

$$= e_{(X_1 \otimes 1 - 1 \otimes X_1,\dots, X_n \otimes 1 - 1 \otimes X_n, f_1 \otimes 1, \dots, f_{d-1} \otimes 1, f \otimes 1)}(\hat{B} \hat{\otimes}_k \hat{B}/\hat{p})$$

[4, p. 300], for $X_1 \otimes 1 - 1 \otimes X_1, \dots, X_n \otimes 1 - 1 \otimes X_n$ is a prime sequence in $\hat{B} \hat{\otimes}_k \hat{B}/\hat{p}$ as will be shown below. As will also be shown below, $f_1 \otimes 1, \dots, f_{d-1} \otimes 1, f \otimes 1$ is a prime sequence in $\hat{B} \hat{\otimes}_k \hat{B}/\hat{p}$. The above equality may now be continued.

$$e_{(f_1,\dots,f_{d-1})}(A/Af)$$

$$= e_{(X_1 \otimes 1 - 1 \otimes X_1,\dots, X_n \otimes 1 - 1 \otimes X_n)}(\hat{B}/(f_1, \dots, f_{d-1}, f) \hat{\otimes}_k \hat{B}/\hat{p}) \quad [4, \text{p. } 300]$$

$$= \chi(B/(f_1, \dots, f_{d-1}, f), B/p) \quad [6, \text{p. V-12}]$$

$$= i(Z(B/p) \cdot Z(B/Bf_1) \cdots Z(B/Bf_{d-1}) \cdot Z(B/Bf), m) \quad [6, \text{p. V-20}].$$

It must be shown that $X_1 \otimes 1 - 1 \otimes X_1, \dots, X_n \otimes 1 - 1 \otimes X_n$ *is a prime sequence in*

$$\hat{B} \hat{\otimes}_k \hat{A} \simeq (\cdots ((\hat{A}[[X_1]])\, [[X_2]]) \cdots)[[X_n]].$$

By induction, it follows from the fact that $X_1 - \alpha$ is a regular element of $R[[X_1]]$ for any $\alpha \in R$ where R is a noetherian ring.

It must also be shown that $f \otimes 1, f_1 \otimes 1, \dots, f_{d-1} \otimes 1$ *is a prime sequence* in $\hat{B} \hat{\otimes}_k \hat{A}$. $(f, f_1, \dots, f_{d-1})$ has height d in B, so $f, f_1, \dots, f_{d-1}$ is a prime sequence in B. Let R and S be two rings containing as a subring the field k, and let α be a regular element of R. $0 \longrightarrow R \xrightarrow{m_\alpha} R$ is exact where m_α denotes multiplication by α. S is k-flat, $0 \longrightarrow R \otimes_k S \xrightarrow{m_\alpha \otimes_k S} R \otimes_k S$ is exact, and $\alpha \otimes 1$ is a regular element of $R \otimes_k S$. It follows immediately that $f \otimes 1, f_1 \otimes 1, \dots, f_{d-1} \otimes 1$ is a prime sequence of $B \otimes_k A$. If R is a Zariski ring and if $\hat{R}$ is the completion of R, then $f_1, \dots, f_d$ is a prime sequence in R if and only if $f_1, \dots, f_d$ is a prime sequence in $\hat{R}$ [7, Chapter VIII, §5]. A and B are affine over k, so $B \otimes_k A$ is noetherian, and $B \otimes_k A$ is a Zariski ring with completion $\hat{B} \hat{\otimes}_k \hat{A}$. Thus $f \otimes 1, f_1 \otimes 1, \dots, f_{d-1} \otimes 1$ is a prime sequence in $\hat{B} \hat{\otimes}_k \hat{A}$.

2. **The behavior of μ_A under reduction of A and integral extension of A.** Let A be a *nonimbedded* local ring (the associated prime ideals of (0) in A are all

minimal). Let IA be the integral closure of A contained in QA, the total quotient ring of A. The minimal (height zero) prime ideals of A, IA and QA are in a bijective correspondence. Let N be a minimal prime ideal of A. Then $\lambda_N(0) = \lambda_{(IA)N}(0) = \lambda_{(QA)N}(0)$, and $I(A/N) \simeq IA/IN$ where $IN = (IA)N$. $IA \simeq A'_1 \oplus \cdots \oplus A'_n$ where $I(A'_i) = A'_i$ and A'_i has a unique minimal prime ideal N'_i.

$$A'_1 \oplus \cdots \oplus A'_{i-1} \oplus N'_i \oplus A'_{i+1} \oplus \cdots \oplus A'_n = IN_i$$

for $i = 1, \ldots, n$ are the minimal prime ideals of IA. Thus a maximal ideal of IA contains a unique minimal prime ideal.

LEMMA 1. *Let A be a dimension one nonimbedded local ring with maximal ideal m. Let IA be the integral closure of A in its total quotient ring QA. There are only a finite number of prime ideals $m_1, \ldots, m_s$ of IA lying over m, and the indices $[IA/m_i: A/m]$ are finite for $i = 1, \ldots, s$. Let $A_i = (IA)_{m_i}$. If f is an element of A,*

$$l_A(A/Af) = \sum_{i=1,\ldots,s} n_i \lambda_{N_i}(0)\ [IA/m_i: A/m]\ l_{A_i}(A_i/A_i f)$$

the n_i being positive integers depending only upon A/N where N is the nil radical of A.

If IA/IN is a noetherian A-module, then $n_i = 1$ for $i = 1, \ldots, s$. The n_i may be greater than one, for in Nagata's example [3, E 3.2, p. 206], $s = 1$ and $n_1 = p$.

PROOF. It may be assumed that f is a regular element of A, for otherwise both sides of the equality are infinite. Let B be a finite A-submodule of IA, and let $a \in A$ be regular and such that $aB \subset A$.

$$\begin{aligned} l_A(B/Bf) &= l_A(Ba/Baf) = l_A(A/Aaf) - l_A(A/Ba) - l_A(Baf/Aaf) \\ &= l_A(A/Af) + l_A(A/Aa) - l_A(A/Ba) - l_A(Ba/Aa) \\ &= l_A(A/Af). \end{aligned}$$

By [3, Theorem 21.2, p. 70], or by the first part of the proof of [7, Theorem 24, p. 297],

$$l_A(A/Af) = \sum_{i=1,\ldots,s_B} [B/p_i : A/m]\ l_B(B_{p_i}/B_{p_i} f)$$

where $p_1, \ldots, p_{s_B}$ are the prime ideals of B lying over m. There are a finite number of prime ideals in IA lying over m, for $s_B \leq l_A(A/Af)$. Let $m_1, \ldots, m_s$ be the maximal ideals of IA. Note that

$$l_A(\text{dir lim}_\iota M_\iota) \leq \max_\iota \{l_A(M_\iota)\},$$

$IA/m_i = \text{dir lim}_B B/B \cap m_i$ and $[IA/m_i: A/m]$ is finite.

Let $\alpha_i \in IA$ be such that $\alpha_i \in m_i$ and $\alpha_i \notin \bigcup_{j \neq i} m_j$. Let $\beta_1, \ldots, \beta_t \in IA$ be such that

$$[A[\beta_1, \ldots, \beta_t]/(m_i \cap A[\beta_1, \ldots, \beta_t]) : A/m] = [IA/m_i : A/m]$$

for $i = 1, \ldots, s$. Let $A' = A[\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t]$. By the formula above, letting A be $A'_{A' \cap m_i}$, it can be assumed that $s = 1$ and $[IA/m_i : A/m] = 1$. Then for a finite extension $B \subset IA$ of A, $l_A(A/Af) = l_B(B/Bf)$. The nil radical N of A is now a prime ideal.

First assume that $I(A/N)$ is a noetherian A/N-module. By a finite extension of A in IA it can be assumed that A/N is normal, and thus that A/N is a regular local ring of dimension one [3, Theorem 33.2, p. 115 and Theorem 21.4, p. 40]. Let $x \in m/N$ generate m/N in A/N. Let

$$(0) = N_0 \subset N_1 \subset \cdots \subset N_{t-1} = NA_N \subset N_t = A_N$$

be a composition series of A_N over A_N, and let $n_i = A \cap N_i$. n_i/n_{i-1} is a principal A/N-module: If $\alpha_1, \ldots, \alpha_q \in n_i/n_{i-1}$ are nonzero and generate n_i/n_{i-1} as an A or A/N-module, there are $v, v_j \in A \sim N$ such that $v_j\alpha_j = v\alpha_1$ for $j = 1, \ldots, q$ (for there is a bijective correspondence between the ideals of A_N and their contractions in A). Viewed as A/N-modules, $\alpha_j = u_j x^{t_j} \alpha_1$ where u_j is a unit in A/N and where t_j is an integer. Let $t_k = \min\{t_1, \ldots, t_q\}$. $n_i/n_{i-1} = A\alpha_k$. So there are $a_1, \ldots, a_t \in N$ with $n_i = (a_1, \ldots, a_i)$. For $i = 1, \ldots, t$,

$$0 \longrightarrow \frac{n_i + Af}{n_{i-1} + Af} \longrightarrow \frac{A}{n_{i-1} + Af} \longrightarrow \frac{A}{n_i + Af} \longrightarrow 0$$

is exact. Map $A \longrightarrow (n_i + Af)/(n_{i-1} + Af)$ by $y \longmapsto ya_i + (f, a_1, \ldots, a_{i-1})$. Suppose $ya_i \in (f, a_1, \ldots, a_{i-1})$. There are $c, c_1, \ldots, c_{i-1} \in A$ such that $cf = c_1a_1 + \cdots + c_{i-1}a_{i-1} - ya_i$. $y \notin N$ and n_i is N-primary because it is the contraction of an $A_N N$-primary ideal, so $c \in (a_1, \ldots, a_i)$. Thus there is an element b of A such that $ya_i - ba_i f \in (a_1, \ldots, a_{i-1})$. $a_i \notin (a_1, \ldots, a_{i-1})$ which is N-primary, so $y - bf \in N$. Hence

$$(n_i + Af)/(n_{i-1} + Af) \simeq A/(N + Af),$$

and

$$l_A(A/Af) = \lambda_N(0)\, l_{A/N}(A/(N + Af)) = \lambda_N(0)\, l_{IA/IN}(IA/IA \cdot f).$$

Now drop the assumption that $I(A/N)$ is a finite A/N-module. Let $\hat{A}$ be the m-adic completion of A. $l_A(A/Af) = l_{\hat{A}}(\hat{A}/\hat{A}f)$. The pair A, m is a Zariski ring, so $(A/N)^{\wedge} \simeq \hat{A}/\hat{N}$, $\hat{A}$ and $\hat{N}$ are unmixed [7, Chapter VIII, §4]. Letting M_j be a minimal prime ideal of $\hat{A}$, $I(\hat{A}/M_j)$ is a finite $\hat{A}/M_j$-module [3, Theorem 32.1, p. 112]. By the *finite case* above

$$l_{\hat{A}}(\hat{A}/\hat{A}f) = \sum_j \lambda_{M_j}(0)\, l_{\hat{A}/M_j}((\hat{A}/M_j)/(\hat{A}/M_j)f).$$

$A \subset A_N \subset \hat{A}_{M_j}$ canonically. Let

$$(0) = N_0 \subset N_1 \subset \cdots \subset N_{t-1} = A_N N \subset N_t = A_N$$

be a composition series of A_N. $N_i \otimes_{A_N} \hat{A}_{M_j}$ can be refined into a composition series for A_{M_j}. Now $N_i/N_{i-1} \simeq A_N/A_N N$, this completion and localization are exact, so $N_i/N_{i-1} \otimes_{A_N} A_{M_j}$ are all isomorphic for $i = 1, \ldots, t$ of length

$$\lambda_{M_j/\hat{N}}(0) = l_{(\hat{A}/\hat{N})_{M_j/\hat{N}}}((\hat{A}/\hat{N})_{M_j/\hat{N}}),$$

and $\lambda_{M_j}(0) = \lambda_N(0)\lambda_{M_j/\hat{N}}(0)$. Thus

$$l_{\hat{A}}(\hat{A}/\hat{A}f) = \lambda_N(0) l_{\hat{A}/\hat{N}}((\hat{A}/\hat{N})/(\hat{A}/\hat{N})f),$$

and it follows that

$$l_A(A/Af) = \lambda_N(0)\, l_{A/N}(A/(N + Af)).$$

$I(A/N) \simeq IA/IN$, and IA/IN is a regular local ring of dimension one [3, Theorem 33.2, p. 115 and Theorem 12.4, p. 40]. Let x be a generator of the maximal ideal m_1 of IA and let u be a unit in IA such that for some integer n, $f = ux^n$. By a finite extension of A it may be assumed that u and x are elements of A. To finish the proof, notice that $l_{IA}(IA/(IA)x) = 1$ and $IN \subset (IA)x$ so that

$$\frac{l_{A/N}((A/N)/(A/N)f)}{l_{IA}(IA/(IA)f)} = l_{A/N}((A/N)/(A/N)x).$$

Let $n_1 = l_{A/N}((A/N)/(A/N)x)$.

LEMMA 2. *Let A be a local ring with maximal ideal m, let $N_1, \ldots, N_n$ be the prime ideals of A of depth $d = \dim A$. For every regular element f of A*

$$\mu_A(f) = \sum_{i=1,\ldots,n} \lambda_{N_i}(0)\, \mu_{A/N_i}(f + N_i).$$

PROOF. If $\dim A = 0$, the formula holds trivially. Let p be a prime ideal of A of depth $d - 1$ and containing f. Then $B = A_p$ is of dimension one and is nonimbedded, for f is a regular element. Note that if $N_i \subset p$, then $\lambda_{N_i}(0) = \lambda_{BN_i}(0)$. By Lemma 1, applied to B and to B/BN_i for $N_i \subset p$,

$$l_B(B/Bf) = \sum_{N_i \subset p} \lambda_{N_i}(0) l_{B/BN_i}((B/BN_i)/(B/BN_i)f),$$

and by [6, p. V-3],

$$\mu_A(f) = \sum_p l_p(A/Af)\, e_m(A/p)$$
$$= \sum_p \sum_{N_i \subset p} \lambda_{N_i}(0)\, l_{p/N_i}((A/N_i)/(A/N_i)f) e_m(A/p)$$
$$= \sum_{i=1,\ldots,n} \lambda_{N_i}(0)\, \mu_{A/N_i}(f + N_i).$$

LEMMA 3. *Let A be a dimension one local ring with maximal ideal m, let $m_1, \ldots, m_s$ be the prime ideals of IA lying over m, and let $A_i = IA_{m_i}$. For every regular element f of A,*

$$\mu_A(f) = \sum_{i=1,\ldots,s} \lambda_{N_i}(0) n_i [IA/m_i : A/m] \mu_{A_i}(f)$$

for some positive integers $n_1, \ldots, n_s$ where N_i is the minimal prime ideal of A_i.

This is a restatement of Lemma 1. (If A is imbedded, the only regular elements of A are the units, and the formula holds trivially.)

REMARK. *Lemma* 3 *does not necessarily hold if the dimension of A is greater than one.* Let

$$A = k[w, x, y, z]_{(w,x,y,z)} = k[W, X, Y, Z]_{(W,X,Y,Z)}/(X^2 - Z^3, XY - W^3)$$

where k is a field. By direct computation $\mu_A(x) = 9$ and $\mu_A(y) = 6$.

$$A \simeq k[ts, t^3, s^3, t^2]_{(ts,t^3,s^3,t^2)} \subset k[s, t]_{(s,t)}$$

where s and t are independent transcendentals over k, and $IA \simeq k[s, t]_{(s,t)}$. Thus $\mu_{IA}(x) = \mu_{IA}(y) = 3$. By the Corollary of Proposition 2, $\mu_{IA} = v$ where v is the order valuation of $k[s, t]_{(s,t)}$ having valuation ring $k(s/t)[t]_{(t)}$. $\mu_A = v + w$ where w is the valuation having valuation ring $k(t/s^2)[s]_{(s)}$. (See §4.)

3. The first neighborhood ring of A: a quadratic transform of A which is compatible with μ_A. Let $\mathcal{G}_m A$ be the associated graded ring of A with respect to m. Let $m = (x_1, \ldots, x_n)$. The natural homomorphisms

$$A[X_1, \ldots, X_n] \longrightarrow k[X_1, \ldots, X_n] \longrightarrow \mathcal{G}_m A$$

(where $k = A/m$) will be used. Let $A[X]$ denote $A[X_1, \ldots, X_n]$, and let $k[X]$ denote $k[X_1, \ldots, X_n]$. I will denote the ideal $(X_1, \ldots, X_n)$ of $A[X]$, $k[X]$, and $\mathcal{G}_m A$.

A familiarity with Northcott's *The neighborhoods of a local ring* [5] is assumed. For the definition of the first neighborhood ring $\mathfrak{R}$ of A, see [5, p. 361]. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height one prime ideals of $\mathfrak{R}$ lying over m, and let p_i be the prime ideal of $\mathcal{G}_m A$ corresponding to $\mathfrak{p}_i$ [5, Propositions 1–4]. The preimage of p_i in $k[X]$ will also be denoted by p_i. For the definition of a superficial element of A see [5, p. 362], [3, p. 72 and Theorem 30.1, p. 103], or [7, p. 285].

LEMMA 4. *Let A be an entire local ring with maximal ideal m and an infinite residue field k. Let $\mathfrak{R}$ be the first neighborhood ring of A, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height one prime ideals of $\mathfrak{R}$ lying over m, let $\mathfrak{R}_i = \mathfrak{R}_{\mathfrak{p}_i}$, and let p_i be the prime ideal of $\mathcal{G}_m A$ corresponding to $\mathfrak{p}_i$. Then*

$$\mu_A(f) = e_I(\mathcal{G}_m A/p_1)\mu_{\mathfrak{R}_1}(f) + \cdots + e_I(\mathcal{G}_m A/p_r)\mu_{\mathfrak{R}_r}(f)$$

for all $f \in A$.

PROOF. *The equality is easily shown to hold for a superficial element of A.* Let $f \in A$ be superficial of degree s. $\mu_A(f) = e_m(A/Af) = se_m(A)$ [7, Lemma 4, p. 286], and

$$\mu_A(f) = s(e_I(k[X]/p_1)e_{\mathfrak{p}_1}(\mathfrak{R}_1/\mathfrak{R}_1 m) + \cdots + e_I(k[X]/p_r)e_{\mathfrak{p}_r}(\mathfrak{R}_r/\mathfrak{R}_r m))$$

[5, formula E, p. 370]. Let x be a superficial element of A of degree one. $f/x^s \in \mathfrak{R}_i$, $\mathfrak{R}_i m = \mathfrak{R}_i x$ for $i = 1, \ldots, r$, and

$$\begin{aligned}\mu_A(f) &= s(e_I(k[X]/p_1)\mu_{\mathfrak{R}_1}(x) + \cdots + e_I(k[X]/p_r)\mu_{\mathfrak{R}_r}(x))\\ &= e_I(k[X]/p_1)\mu_{\mathfrak{R}_1}(f) + \cdots + e_I(k[X]/p_r)\mu_{\mathfrak{R}_r}(f).\end{aligned}$$

The proof of the equality in general will occupy the rest of this section.

First let $\dim A \geqslant 2$. The proof will proceed by fixing the element $f \in A$ and blowing up A to a one-dimensional ring B such that $\mathfrak{R}^1 = \mathfrak{R}_1 \cap \cdots \cap \mathfrak{R}_r$ is an integral extension of B and such that $\mathcal{G}_{mB}(B/Bf)$ is nearly a linear section of $\mathcal{G}_m(A/Af)$.

Let v_A be the order function of A with respect to m. Let x be a superficial element of A of degree one, let $m = (x_1, \ldots, x_n)$ and let Π be a form of degree one in $A[X_1, \ldots, X_n]$ with $x = \Pi(x_1, \ldots, x_n)$. Π will also denote its image modulo m in $k[X_1, \ldots, X_n]$. Consider the diagram,

$$\begin{array}{ccc} A[X_1, \ldots, X_n] & \xrightarrow[\rho]{\hspace{2em}} & k[X_1, \ldots, X_n] \\ \big\downarrow \chi & & \big\downarrow \psi \\ A & \xrightarrow[\sigma]{\hspace{2em}} & \mathcal{G}_m A \end{array}$$

where $\sigma(g) = (g + m^{v_A(g)+1})/m^{v_A(g)+1}$, ψ is the canonical homomorphism and $k = A/m$, χ is the homomorphism with $\chi(X_i) = x_i$ and $\chi|_A = \mathrm{id}_A$, and $\rho(F)$ is the leading form modulo m of F. $\sigma(Af)$ is an ideal of $\mathcal{G}_m A$, but σ need not be a homomorphism. Let $\tau Af = \psi^{-1}\sigma(Af)$, let $\omega Af = \chi^{-1}(Af) = (X_1 - x_1, \ldots, X_n - x_n, f)$, and let σAf denote $\sigma(Af)$.

$\rho(\omega Af) = \tau Af$. First notice that if $E \in \omega Af$ and $\deg E = v_A(\chi E) = s$ then $\psi\rho E = \psi(E + m[X] + I^{s+1}) = E(x_1, \ldots, x_n) + m^{s+1}$. Secondly notice that $\psi^{-1}(0) = \tau A0 \subset \rho(\omega Af)$. If $E \in \omega Af$ and if $\psi\rho E = 0$ then $\rho E \in \psi^{-1}(0) \subset$

$\rho(\omega Af)$. If $E \in \omega Af$ and if $\psi\rho E \neq 0$ then $\deg E = v_A(\chi E)$, $\psi\rho E = \sigma\chi E$, and $\rho E \in \tau AF$. Hence $\rho(\omega Af) \subset \tau Af$. Let $e \in Af$. Let $E \in \omega Af$ be such that $\deg E = v_A(e)$ and $\chi E = e$. Then $\sigma e = \psi\rho E$, $\rho E \in \psi^{-1}(\sigma e)$, and $\tau Af \subset \rho(\omega Af)$.

Let $\mathfrak{p}$ be an isolated prime ideal of $\tau A0$. Then depth $\mathfrak{p} = \dim A -$ height $\mathfrak{p} \geqslant 2$ and depth$(\mathfrak{p}, \Pi) \geqslant 1$.

Choose Θ to be a form of degree one in $A[X] = A[X_1, \ldots, X_n]$ such that $y = \Theta(x_i)$ is a superficial element of A and a superficial element of A/Af, such that Θ is contained in no isolated prime ideal of $(\mathfrak{p}, \Pi)$ for any isolated prime ideal $\mathfrak{p}$ of $\tau A0$, and such that y is contained in no associated prime ideal of Ax other than possibly m. Each condition is viewed as a condition on form ideals in $k[X]$. Let Θ also denote its image modulo m in $k[X]$.

Let $u = y/x$. Let P be the kernel of the canonical homomorphism of $A[U]$ onto $A[u]$ where $A[U]$ is the polynomial ring in one variable and U maps to u. $P \cap A = (0)$, and it follows that P is of height one in $A[U]$. Letting $\mathcal{D}_A$ denote the set of prime ideals of A which occur as an imbedded prime ideal of a proper principal ideal of A (see [2, §6]), $Q \in \mathcal{D}_{A[U]}$ if and only if $Q \cap A \in \mathcal{D}_A$ and $Q = (Q \cap A) \cdot A[U]$. $y - xU$ is prime in $A[U]$ if and only if x, y form a prime sequence in A, but this is the case if and only if $m \notin \mathcal{D}_A$. If $m \notin \mathcal{D}_A$ then $P = (y - xU)$, and $P \subset m[U]$. If $m \in \mathcal{D}_A$ then P and $m[U]$ are the associated prime ideals of $(y - xU)$. For if Q is an associated prime ideal of $(y - xU)$ of height greater than one then $x, y \in Q \cap A$ and $Q = m[U]$. If Q is of height one, either $Q \cap A = q \neq (0)$, in which case $Q = q[U]$ and $x, y \in q$ which contradicts the choice of y, or $Q \cap A = (0)$ in which case $Q = (QA)[U] \cdot (y - xU) = P$. It again follows that $P \subset m[U]$. So $A[u]/m[u] \simeq k[U]$, and $\bar{u} = u + m \cdot A[u]$ is transcendental over k.

Let $S = A[u] \sim mA[u]$ and let $B = S^{-1}A[u]$. $B/mB \simeq k(\bar{u})$ a simple transcendental extension of k. Dim $A[U] = \dim A + 1$, the kernel P of the homomorphism $A[U] \longrightarrow A[u]$ is height one, $m[U]$ is of height equal to $\dim A$, and $\dim B = \dim A - 1$. Consider $G_{mB}B$ and the commutative diagram

$$\begin{array}{ccc}
A[X_1, \ldots, X_n] & \hookrightarrow & B[X_1, \ldots, X_n] \\
\downarrow \rho & & \downarrow \rho \\
k[X_1, \ldots, X_n] & \hookrightarrow & k(\bar{u})\,[X_1, \ldots, X_n] \\
\downarrow \psi & & \downarrow \psi \\
G_m A & \xrightarrow[\phi]{} & G_{mB}B
\end{array}$$

where ϕ is the canonical homomorphism induced by the inclusion $A \subset B$. Define σ, τ and ω for B as was done for A. Notice that $\omega Af \subset \omega Bf$, so $\tau Af \subset \tau Bf$. $\Theta - u\Pi \in \omega Bf$. Let q be an associated prime ideal of τAf which is not $I = (X_1, \ldots, X_n)$. If $\Theta - \bar{u}\Pi \in k(\bar{u}) \cdot q$, then $\Theta - \bar{u}\Pi \in k[\bar{u}] \cdot q$ and $\Theta \in q$, which is a contradiction to the superficiality of y. Therefore $\Theta - \bar{u}\Pi \notin k(\bar{u})q$, and $\Theta - \bar{u}\Pi$ *is superficial as an element of* $k(\bar{u})[X]/k(\bar{u}) \cdot \tau Af$.

Now $\mu_A(f) = e_I(k[X]/\tau Af)$ and $\mu_B(f) = e_I(k(\bar{u})[X]/\tau Bf)$. These modules are homogeneous and their lengths over $k[X]$ or $k(\bar{u})[X]$ are their dimensions over k or $k(\bar{u})$. Thus $\mu_A(f) = e_I(k(\bar{u})\,[X]/k(\bar{u}) \cdot \tau Af)$. By Lemmas 3 and 4 of [7, pp. 285–286], if $\dim A > 2$,

$$e_I(k(\bar{u})[X]/k(\bar{u}) \cdot \tau Af) = e_I(k(\bar{u})[X]/(\tau Af, \Theta - \bar{u}\Pi)),$$

and if $\dim A = 2$,

$$e_I(k(\bar{u})[X]/k(\bar{u}) \cdot \tau Af) = e_I(k(\bar{u})[X]/(\tau Af, \Theta - \bar{u}\Pi)) - l_{k(\bar{u})[X]}(I^c + ((I^n, \tau Af) \colon \Theta - \bar{u}\Pi)/(I^c, \tau Af))$$

for all large enough n and c with $n > c$. Because $\Theta - \bar{u}\Pi$ is contained in no associated prime ideal of $k(\bar{u}) \cdot \tau Af$ other than possibly I, the homogeneous parts of like degree of $k(\bar{u}) \cdot \tau Af$ and of $(k(\bar{u}) \cdot \tau Af \colon \Theta - \bar{u}\Pi)$ are equal for sufficiently large degree. So for large enough n and c, over $k(\bar{u})$

$$(I^c + ((I^n, \tau Af) \colon \Theta - \bar{u}\Pi)/(I^c, \tau Af)) \simeq (k(\bar{u}) \cdot \tau Af \colon \Theta - \bar{u}\Pi)/k(\bar{u}) \cdot \tau Af,$$

and for $\dim A = 2$,

$$e_I(k(\bar{u})[X]/k(\bar{u}) \cdot \tau Af) = e_I(k(\bar{u})[X]/(\tau Af, \Theta - \bar{u}\Pi)) - \dim_{k(\bar{u})}(k(\bar{u}) \cdot \tau Af \colon \Theta - \bar{u}\Pi)/k(\bar{u}) \cdot \tau Af.$$

Let

$$\alpha = \dim_{k(\bar{u})} \tau Bf/(\tau Af, \Theta - \bar{u}\Pi)$$

and

$$\beta = \dim_{k(\bar{u})}(k(\bar{u}) \cdot \tau Af \colon \Theta - \bar{u}\Pi)/k(\bar{u}) \cdot \tau Af.$$

It is to be shown that $\alpha = \beta$. Then α is finite, for β is finite by the superficiality of $\Theta - \bar{u}\Pi$, and it follows that if $\dim A > 2$, $\mu_A(f) = \mu_B(f)$. If $\dim A = 2$ it follows from $\alpha = \beta$ that $\mu_A(f) = \mu_B(f)$.

If $\mathfrak{A}$ is a set of polynomials in $X_1, \ldots, X_n$, let $\mathfrak{A}_{(d)}$ be the set of all elements of $\mathfrak{A}$ which have no nonzero homogeneous component of degree strictly less than d, and let $\mathfrak{A}_d$ be the set of all homogeneous elements of $\mathfrak{A}$ of degree d.

Let $S = A[U] \sim m[U]$, and let $A(U)$ denote $S^{-1}A[U]$. Let $\tau(P, f) = \rho(P, \omega A(U)f)$ and $\tau(\Theta - U\Pi, f) = \rho(\Theta - U\Pi, \omega A(U)f)$. Consider

$$\begin{array}{ccc} A(U)[X] & \xrightarrow[\rho]{} & k(U)[X] \\ \downarrow \psi & & \downarrow \bar{\psi} \\ B[X] & \xrightarrow[\rho]{} & k(\bar{u})[X] \end{array}$$

where $\rho(\alpha)$ is the leading form in $X_1, \ldots, X_n$ of α modulo $mA(U)[X]$ or $mB[X]$, where $\psi(U) = u$ and $\psi|_{A[X]} = \mathrm{id}_{A[X]}$, and where $\bar{\psi}(U) = \bar{u}$ and $\bar{\psi}|_{k[X]} = \mathrm{id}_{k[X]}$. Because $P \subset (P, \omega A(U)f)$,

$$\bar{\psi}\tau(P, f) = \rho\psi(P, \omega A(U)f) = \tau Bf.$$

Note that $\bar{\psi}: k(U)[X] \longrightarrow k(\bar{u})[X]$ is an isomorphism over the isomorphism $k(U) \simeq k(\bar{u})$ induced by $\bar{\psi}$. Let

$$\gamma = \dim_{k(U)}\tau(P, f)/\tau(\Theta - U\Pi, f) = \dim_{k(\bar{u})}\tau Bf/\bar{\psi}\lambda(\Theta - U\Pi, f).$$

Then

$$\dim_{k(U)}\tau(f, \Theta - U\Pi)/(\tau Af, \Theta - U\Pi) = \alpha - \gamma.$$

Let H be $\rho((\omega A(U)f)\hat{}:_{A(U)[[X]]} \Theta - U\Pi)$ where $\hat{}$ denotes the I-adic completion. Let Q be an associated prime ideal of $\omega A(U)f$. $(X_1 - x_1, \ldots, X_n - x_n) \subset Q$, so $Q \subset (mA(U), I)$. $A(U)[X]_{(mA(U),I)}$ with the I-adic topology is a Zariski ring with completion $A(U)[[X]]$. Hence

$$((\omega A(U)f)\hat{}:_{A(U)[[X]]}\Theta - U\Pi) = (\omega A(U)f:_{A(U)[X]}\Theta - U\Pi)\hat{}$$

[7, Corollary 4, p. 266], and $H = p(\omega A(U)f: \Theta - U\Pi)$. So $\bar{\psi}H \subset (k(\bar{u}) \cdot \tau Af: \Theta - U\Pi)$. Let

$$\delta = \dim_{k(U)}H/k(U) \cdot \tau Af.$$

Then

$$\dim_{k(U)}(k(U) \cdot \tau Af: \Theta - U\Pi)/H = \beta - \delta.$$

It is to be first shown that $\alpha - \gamma = \beta - \delta$.

Let $M \in A(U)[X_1, \ldots, X_n]$ be homogeneous of degree d such that $M + mA(U)[X] \in \tau(\Theta - U\Pi, f)$. The following four assertions follow easily from the fact that $x_i - X_i \in \omega A(U)f$. There is an integer $h \leqslant d - 1$ and forms $H_i \in A(U)[X]$ of degree $i = h, \ldots, d - 1$ such that

$$(\Theta - U\Pi)(H_h + \cdots + H_{d-1}) + M \in \omega A(U)f + A(U)[X]_{(d+1)}.$$

If $M - M' \in mA(U)[X]_d$, then

$$(\Theta - U\Pi)(H_h + \cdots + H_{d-1}) + M' \in \omega A(U)f + A(U)[X]_{(d+1)}.$$

If $H_h - H'_h \in mA(U)[X]_h$, there are forms $H'_i \in A(U)[X]$ for $i = h + 1, \ldots, d - 1$ such that

$$(\Theta - U\Pi)(H'_h + \cdots + H'_{d-1}) + M \in \omega A(U)f + A(U)[X]_{(d+1)}.$$

If $F \in A(U)[X]_d$ and if $F + mA(U)[X] \in k[X] \cdot \tau Af$, then

$$(\Theta - U\Pi)(H_h + \cdots + H_{d-1}) + (M + F) \in \omega A(U)[X] + A(U)[X]_{(d+1)}.$$

Note that $H_h + mA(U)[X] \in (k(U) \cdot \tau Af : \Theta - U\Pi)$. Let $h(M) < \deg M$ be the maximal degree of all such H_h as above. Let $H(M)$ be the set of all such H_h as above with $h = h(M)$. $M + mA(U)[X] \in (\tau Af, \Theta - U\Pi)$ if and only if $h(M) = \deg M - 1$ which is true if and only if $\mathcal{H}_{h(M)} \subset H(M)$ (which in this case is $A(U)[X]_{h(M)}$). If $b \in A(U) \sim mA(U)$, $bH(M) = H(bM)$. If $H \in H(M)$ then

$$(H + mA(U)[X]_{h(M)}) + \mathcal{H}_{h(M)} \subset H(M)/mA(U)[X]_{h(M)},$$

and $H(M)$ will be considered as a subset of $(k(U) \cdot \tau Af : \Theta - U\Pi)/\mathcal{H}$.

A $k(U)$-linear injection of $\tau(f, \Theta - U\Pi)/(\tau Af, \Theta - U\Pi)$ into $(k(U) \cdot \tau Af : \Theta - U\Pi)/\mathcal{H}$ is to be defined. Let $M_1, \ldots, M_a \in A(U)[X]$ be forms such that their residues modulo $mA(U)[X]$ are in $\tau(f, \Theta - U\Pi)$, such that their residues in $\tau(f, \Theta - U\Pi)/(\tau Af, \Theta - U\Pi)$ are linearly independent over $k(U)$, such that $h(M_i) \leqslant h(M_{i+1})$ and such that if $h(M_i) = h(M_{i+1})$ then $\deg M_i \geqslant \deg M_{i+1}$. Choose $\eta_i \in H(M_i)$. Suppose $\eta_i, \ldots, \eta_{t-1}$ are linearly independent over $k(U)$, and suppose $\eta_t = \bar{\alpha}_1\eta_1 + \cdots + \bar{\alpha}_{t-1}\eta_{t-1}$ where $\alpha_i \in A(U)$. The $\bar{\alpha}_i$ are nonzero only for those M_i with $h(M_i) = h(M_t)$. $h(M_t) = h(M_{t-1})$, for $\eta_t \neq 0$. Let $M_s, \ldots, M_{t-1}$ be exactly those M_i with $i < t$, $h(M_i) = h(M_t)$ and $\deg M_i = \deg M_t$. Then $h(M_t - \alpha_s M_s - \cdots - \alpha_{t-1}M_{t-1}) > h(M_t)$, so replace M_t by $M_t - \alpha_s M_s - \cdots - \alpha_{t-1}M_{t-1}$, choose a new η_t, and reorder $M_t, \ldots, M_a$. With a finite number of repetitions of the above process $\eta_1, \ldots, \eta_t$ will be linearly independent, for at worst $h(M_t)$ will eventually be greater than $h(M_{t-1})$, and linear independence will follow. Thus $a \leqslant \beta - \delta$, and $\alpha - \gamma \leqslant \beta - \delta$.

A construction analogous to the above is used to derive the opposite inequality. Let $H \in A(U)[X]_d$ with $H + mA(U)[X] \in (k(U) \cdot \tau Af : \Theta - U\Pi)$. Let $m(H)$ be the maximal integer m such that there exists a form M of degree m and forms H_i of degree $i = d + 1, \ldots, m - 1$ such that

$$(\Theta - U\Pi)(H + H_{d+1} + \cdots + H_{m-1}) + M \in \omega A(U)f + A(U)[X]_{(m+1)}$$

and $M + mA(U)[X] \notin (\tau Af, \Theta - u\Pi)$. If such a maximum does not exist then $H + mA(U)[X] \in \mathcal{H}$, and if $H + mA(U)[X] \notin \mathcal{H}$, then $m(H) \geqslant \deg H + 1$. Let $M(H)$ be the set of all such M of degree $m(H)$. $M(bH) = bM(H)$ for $b \in A(U) \sim mA(U)$. If $M \in M(H)$ then $M + mA(U)[X] \subset M(H)$,

$$M + mA(U)[X]_{m(H)} + (\tau Af, \Theta - U\Pi)_{m(H)} \subset M(H)/mA(U)[X]_{m(H)}$$

and $M + mA(U)[X]_{m(H)} \in \tau(f, \Theta - U\Pi)$. $M(H)$ will be considered as a subset of $\tau(f, \Theta - U\Pi)/(\tau Af, \Theta - U\Pi)$.

Let $H_1, \ldots, H_{\beta-\delta}$ be forms in $mA(U)[X]$ such that their residues modulo $mA(U)[X]$ are in $(k(U) \cdot \tau Af : \Theta - U\Pi)$, such that their residues form a $k(U)$-basis for $(k(U) \cdot \tau Af : \Theta - U\Pi)/\mathcal{H}$, $m(H_i) \leqslant m(H_{i+1})$ and such that if $m(H_i) =$

$m(H_{i+1})$ then $\deg H_i \geqslant \deg H_{i+1}$. Choose $\mu_i \in M(H_i)$. Suppose $\mu_1, \ldots, \mu_{t-1}$ are linearly independent over $k(U)$ and $\mu_t = \bar{\alpha}_1\mu_1 + \cdots + \bar{\alpha}_{t-1}\mu_{t-1}$ where $\alpha_i \in A(U)$. $\bar{\alpha}_i$ is nonzero only if $m(H_i) = m(H_t)$, $m(H_{t-1}) = m(H_t)$ for $\mu_t \neq 0$, and let $H_s, \ldots, H_{t-1}$ be those H_i with $i < t$, $m(H_i) = m(H_t)$ and $\deg H_i = \deg H_t$. Then $m(H_t - \alpha_s H_s - \cdots - \alpha_{t-1}H_{t-1}) > m(H_t)$. Replace H_t by $H_t - \alpha_s H_s - \cdots - \alpha_{t-1}H_{t-1}$, choose μ_t anew, reorder $H_1, \ldots, H_{\beta-\delta}$, with a finite number of repetitions the injection is defined, and $\alpha - \gamma \geqslant \beta - \delta$.

Thus $\alpha - \gamma = \beta - \delta$. *The final goal in the proof of $\alpha = \beta$ is to show that γ and δ are equal.*

Let $\mathfrak{A} \subset \mathfrak{B}$ be two ideals of $A(U)$. As either $k(U)$ or $A(U)$-modules, $\tau\mathfrak{B}/\tau\mathfrak{A} \simeq \sigma\mathfrak{B}/\sigma\mathfrak{A}$. Now

$$\sigma\mathfrak{B}/\sigma\mathfrak{A} \simeq \sum_{n\geqslant 0} \oplus \frac{(m^n \cap \mathfrak{B} + m^{n+1}/m^{n+1})}{(m^n \cap \mathfrak{A} + m^{n+1}/m^{n+1})}$$

$$\simeq \sum_{n\geqslant 0} \oplus \frac{(m^n \cap \mathfrak{B} + m^{n+1})}{(m^n \cap \mathfrak{A} + m^{n+1})} \simeq \sum_{n\geqslant 0} \oplus \frac{m^n \cap \mathfrak{B}}{(m^n \cap \mathfrak{A} + m^{n+1} \cap \mathfrak{B})}$$

(for $(m^n \cap \mathfrak{B}) \cap (m^n \cap \mathfrak{A} + m^{n+1}) = m^n \cap \mathfrak{A} + m^{n+1} \cap \mathfrak{B}$). Hence, $l_{k(u)}\tau\mathfrak{B}/\tau\mathfrak{A} = l_{A(U)}\mathfrak{B}/\mathfrak{A}$.

So

$$\gamma = l_{A(U)}(P, f)/(y - xU, f),$$

and

$$\delta = l_{A(U)}(A(U)f : y - xU)/A(U)f.$$

Let $\psi \in (A(U)f : y - xU)$. $(\psi/f)(y - xU) \in A(U)$, $f(\psi/f)(y - xU) \in P$, $f \notin P$, so $(\psi/f)(y - xU) \in P$. Let $\xi_1(\psi) = (\psi/f)(y - xU)$. If $\psi \in A(U)f$ then $\xi_1(\psi) \in A(U)(y - xU)$. Hence

$$\xi_1 : (A(U)f : y - xU)/A(U)f \longrightarrow (P, f)/(y - xU, f)$$

is a homomorphism. Let $\psi \in \operatorname{Ker} \xi_1$, that is, let $(\psi/f)(y - xU) = af + b(y - xU)$ for some a and b in $A(U)$. Then $(\psi - bf)(y - xU) = af^2$, and $\psi \in ((A(U)f^2 : y - xU), f)$. If $\phi \in (A(U)f^2 : y - xU)$, then $\phi(y - xU) = af^2$ for some $a \in A(U)$, $\xi_1(\phi) = (\phi/f)(y - xU) = af$, and $\phi \in \operatorname{Ker} \xi_1$. So

$$\operatorname{Ker} \xi_1 = (A(U)f^2 : y - xU), f)/A(U)f.$$

Now,

$$(A(U)f^i : y - xU)/(A(U)f^i : y - xU) \cap A(U)f$$

$$\simeq ((A(U)f^i : y - xU), f)/A(U)f,$$

and a homomorphism

$$\xi_i: (A(U)f^i : y - xU)/(A(U)f^i : y - xU) \cap A(U)f$$
$$\longrightarrow (\cdots(((P, f)/(y - xU, f))/\mathrm{Im}\ \xi_1)/ \ldots)/\mathrm{Im}\ \xi_{i-1}$$

with

$$\mathrm{Ker}\ \xi_i = ((A(U)f^{i-1} : y - xU), f)/A(U)f$$

is to be defined inductively.

If $\psi \in (A(U)f^i\colon\ y - xU)$, let $\xi_i(\psi) = (\psi/f^i)(y - xU) \in P$. If $\psi \in (A(U)f^i\colon y - xU) \cap A(U)f$, then $\psi/f \in (A(U)f^{i-1}: y - xU)$, $\xi_{i-1}(\psi/f) = (\psi/f^i)(y - xU) = \xi_i(\psi)$, and $\xi_i(\psi) \in \mathrm{Im}\ \xi_{i-1}$. Let $\psi \in \mathrm{Ker}\ \xi_i$. Then

$$(\psi/f^i)(y - xU) = af + b(y - xU)$$
$$+ (\psi_1/f)(y - xU) + \cdots + (\psi_{i-1}/f^{i-1})(y - xU)$$

where $\psi_j \in (A(U)f^j : y - xU)$ for $j = 1, \ldots, i - 1$, and

$$(\psi - bf^i - f^{i-1}\psi_1 - \cdots - f\psi_{i-1})(y - xU) = af^{i+1},$$

so $\mathrm{Ker}\ \xi_i \subset ((A(U)f^{i+1}\colon\ y - xU), f)/A(U)f$. If $\phi \in (A(U)f^{i+1} : y - xU)$ then $\xi_i(\phi) = (\phi/f^i)(y - xU) \in A(U)f$, and $\phi \in \mathrm{Ker}\ \xi_i$. Thus

$$\mathrm{Ker}\ \xi_i = ((A(U)f^{i+1}\colon\ y - xU), f)/A(U)f.$$

$\bigcap_i A(U)f^i = (0)$, so $\bigcap_i (A(U)f^{i+1}\colon\ y - xU) = (0)$, and by [3, Theorem 30.1, p. 103], $\bigcap_i \mathrm{Ker}\ \xi_i \subset \bigcap_k (A(U)f + m^k) = A(U)f$. Or by [5, Theorem 1, p. 365], because $y - xU$ is superficial of degree 1, $(m^{i+1}A(U)\colon\ y - xU) = m^i$ for all sufficiently large i, so $\bigcap_i \mathrm{Ker}\ \xi_i \subset \bigcap_i (A(U)f + m^i) = A(U)f$. If $\phi \in P$ there is an integer s such that $f^s\phi \in A(U)(y - xU)$, for there is an integer s such that $P \cap m^s = A(U)(y - xU) \cap m^s$. Then $\xi_s(f^s\phi/(y - xU)) = \phi$.

Let

$$\mathfrak{A}_i = ((A(U)f^i : y - xU), f),$$

and let

$$\mathfrak{B}_i = (\{(\psi/f^i)(y - xU) | \psi \in (A(U)f^i : y - xU)\}, f).$$

Then $\bigcap_i \mathfrak{A}_i = A(U)f$ and $\mathfrak{A}_t = A(U)f$ for some $t \geqslant 1$, for $(A(U)f : y - xU)/A(U)f$ is of finite length. Hence

$$\mathfrak{A}_0 = (A(U)f : y - xU) \supset \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_t = A(U)f,$$

and

$$(y - xU, f) = \mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \cdots \subset \mathfrak{B}_s = (P, f)$$

where $\mathfrak{A}_i/\mathfrak{A}_{i+1} \simeq \mathfrak{B}_{i+1}/\mathfrak{B}_i$ as $A(U)$-modules. Thus $\gamma = \delta$.

The above construction is inductive to dimension one. Let $B_d = A$ and

$B_{d-1} = B$ where d is again the dimension of A, let $\Theta_{d-1} = \Theta$, $y_{d-1} = y$, $u_{d-1} = u$ and $L_{d-1} = \Theta - U\Pi$. Π and $x = \Pi(x_i)$ remain fixed throughout the induction. Suppose B_{j+1} has been defined with the required properties. Let Θ_j be a form of degree one in $A[X]$ such that $y_j = \Theta_j(x_i)$ is a superficial element of B_{j+1} and of $B_{j+1}/B_{j+1}f$, Θ_j is not contained in any associated prime ideal of $(p_i, L_{d-1}, \ldots, L_{j+1})$ other than possibly I nor contained in any isolated prime ideal of $(p_i, L_{d-1}, \ldots, L_{j+1}, \Pi)$ for any isolated prime ideal p_i of $\tau A0$, and such that y_j is contained in no associated prime ideal of $B_{j+1}x$ except possibly mB_{j+1}. The above arguments hold when A is replaced by B_{j+1} and B is replaced by $B_j = S^{-1}B_{j+1}[u_j]$ where $u_j = y_j/x$ and $S = B_{j+1}[u_j] \sim mB_{j+1}[u_j]$.

Let $B = B_1$. B is one dimensional, B is local with maximal ideal mB, and $\mu_A(f) = \mu_B(f)$.

Let $\Re^1$ be $T^{-1}\Re$ where $T = \Re \sim (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r)$ and where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the height one prime ideals of $\Re$. *For every* $i = 1, \ldots, r$,

$$\Re^1\mathfrak{p}_i \cap A[u_{d-1}, \ldots, u_1] = m[u_{d-1}, \ldots, u_1].$$

For let $z \in A[u_{d-1}, \ldots, U_1] \cap \Re^1\mathfrak{p}$ where $\mathfrak{p}$ denotes one of the $\mathfrak{p}_i$. Then $z \in A[u_{d-1}, \ldots, u_1] \cap \mathfrak{p}$. Let p be the prime ideal corresponding to $\mathfrak{p}$ which is associated to $\tau A0$, and let $F(\Theta_{d-1}, \ldots, \Theta_1, \Pi)$ be a form in $\Theta_{d-1}, \ldots, \Theta_1$ and Π with coefficients in A such that

$$F(\Theta_{d-1}(x_i/x), \ldots, \Theta_1(x_i/x), \Pi(x_i/x)) = z.$$

$A[u_{d-1}, \ldots, u_1] \subset \Re$, so $z \in \mathfrak{p}$ and by the correspondence between $\mathfrak{p}$ and p, $F(\Theta_{d-1}, \ldots, \Theta_1, \Pi) + m[X] \in p$. Suppose F modulo m, $\bar{F}$, is nonzero. If $\bar{F}$ were a power of Π, then $\Pi \in p$ which is a contradiction. So there is an integer j such that $d - 1 \geqslant j \geqslant 1, \bar{F} \in k[\Theta_{d-1}, \ldots, \Theta_j, \Pi]$ and $\bar{F} \notin k[\Theta_{d-1}, \ldots, \Theta_{j+1}, \Pi]$. Then

$$\bar{F} = \bar{G}\Pi^e \bmod (\Theta_{d-1} - \Pi, \ldots, \Theta_{j+1} - \Pi) \subset (p, L_{d-1}, \ldots, L_{j+1}, \Pi)$$

for some form $\bar{G} \in k[\Theta_j, \Pi]$ which is not divisible by Π. Letting $s \geqslant 1$ be the degree of $\bar{G}$, $\Theta_j^s \in (p, L_{d-1}, \ldots, L_{j+1}, \Pi)$ which is a contradiction to the choice of Θ_j. Hence $\bar{F} = 0$, and $z \in m[u_{d-1}, \ldots, u_1]$.

B is a ring of fractions of $A[u_{d-1}, \ldots, u_1]$ with $m[u_{d-1}, \ldots, u_1] \subset mB \cap A[u_{d-1}, \ldots, u_1]$. mB is a prime ideal of height one of B, so $mB \cap A[u_{d-1}, \ldots, u_1]$ must be of height one also, and

$$mB \cap A[u_{d-1}, \ldots, u_1] = m[u_{d-1}, \ldots, u_1].$$

It follows that

$$B = A[u_{d-1}, \ldots, u_1]_{m[u_{d-1}, \ldots, u_1]},$$

and therefore $B \subset \Re^1$.

$\mathfrak{R}^1 = \mathfrak{R}_1 \cap \cdots \cap \mathfrak{R}_r$ *is a finite integral extension of* $B = B_1$. The proof is an adaptation of the proof of Theorem 10 [5, p. 371]. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ also denote the proper prime ideals $\mathfrak{R}^1\mathfrak{p}_1, \ldots, \mathfrak{R}^1\mathfrak{p}_r$ of $\mathfrak{R}^1$, let m_i be integers such that $\mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r} \subset \mathfrak{R}^1 m$, and let $\mathfrak{n} = \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r}$. Then $\mathfrak{m}^s \subset (\mathfrak{R}^1 m)^s$ and $(\mathfrak{R}^1 m)^{st} \subset \mathfrak{m}^s$ where $t = \max\{m_1, \ldots, m_r\}$. Let $\hat{B}$ be the mB-adic completion of B, and let $\hat{\mathfrak{R}}$ be the $\mathfrak{R}^1 m$-adic completion of $\mathfrak{R}^1$. $\hat{\mathfrak{R}}$ is a $\hat{B}$-module, $\hat{\mathfrak{R}}$ is the $\mathfrak{m}$-adic completion of $\mathfrak{R}^1$, $\bigcap_{n \geq 0} \mathfrak{m}^n = (0)$, and by [7, Corollary 2, p. 273], the mB-adic topology of B is induced by the $\mathfrak{m}$-adic topology of $\mathfrak{R}^1$. It is clear that $\hat{\mathfrak{R}}/\hat{\mathfrak{R}}m = \mathfrak{R}^1/\mathfrak{R}^1 m$.

$B[x_1/x, \ldots, x_n/x]$ is of dimension one [3, Theorem 33.2, p. 115], and $\mathfrak{R}^1$ is a ring of quotients of $B[x_1/x, \ldots, x_n/x]$. $\mathfrak{p}_j \cap B[x_1/x, \ldots, x_n/x]$ for $j = 1, \ldots, r$ are distinct proper prime ideals of $B[x_1/x, \ldots, x_n/x]$. Let p be a proper prime ideal of $B[x_1/x, \ldots, x_n/x]$. $B[x_1/x, \ldots, x_n/x]$ is a ring of fractions of $A[x_1/x, \ldots, x_n/x]$, so $p \cap A[x_1/x, \ldots, x_n/x]$ is a prime ideal of height one, therefore there is a prime ideal $\mathfrak{p}$ of $\mathfrak{R}^1$ such that $\mathfrak{p} \cap A[x_1/x, \ldots, x_n/x] = p \cap A[x_1/x, \ldots, x_n/x]$, and $\mathfrak{p} \cap B[x_1/x, \ldots, x_n/x] = p$. From the above assertions it is immediate that $\mathfrak{R}^1 = B[x_1/x, \ldots, x_n/x]$.

Let θ_{ji} be the residue of x_i/x modulo $\mathfrak{p}_j$. $\mathfrak{R}^1/\mathfrak{p}_j = k(\bar{u}_1, \ldots, \bar{u}_{d-1})[\theta_{j1}, \ldots, \theta_{jn}]$ is a field, and θ_{ji} are algebraic over $k(\bar{u}) = k(\bar{u}_1, \ldots, \bar{u}_{d-1})$. By multiplying together the m_jth power of a polynomial which modulo $\mathfrak{p}_j$ is the algebraic relation of θ_{ji} over $k(\bar{u})$ for $j = 1, \ldots, r$, there is a relation

$$(x_i/x)^t + \alpha_{t-1}(x_i/x)^{t-1} + \cdots + \alpha_0 \in \mathfrak{R}^1 m$$

where $\alpha_0, \ldots, \alpha_{t-1} \in B$. Therefore $\mathfrak{R}^1/\mathfrak{R}^1 m$ is a finite B/mB module, and $\hat{\mathfrak{R}}$ is a finite $\hat{B}$ module [7, Corollary 2, p. 259]. So for every positive integer s there is a relation

$$\begin{aligned}(x_i/x)^s \in [\hat{B}(x_i/x)^{t-1} + \cdots + \hat{B}(x_i/x) + \hat{B}] \cap B \\ = B(x_i/x)^{t-1} + \cdots + B(x_i/x) + B\end{aligned}$$

for the latter module is finitely generated over the Zariski ring B and is therefore closed. $\mathfrak{R}^1$ is thus finite integral over B.

It is to be shown that $[\mathfrak{R}^1/\mathfrak{p}_s : B/mB] = e_I(k[X]/p_s)$. From the choice of Θ_j it follows that L_j is a superficial element of

$$k(\bar{u}_{d-1}, \ldots, \bar{u}_j)[X]/(p_s, L_{d-1}, \ldots, L_{j+1}),$$

for $\bar{u}_j$ is transcendental over $k(\bar{u}_{d-1}, \ldots, \bar{u}_{j+1})$. The dimensions are greater than one, so

$$e_I(k[X]/p_s) = e_I(k(\bar{u})[X]/(p_s, L_{d-1}, \ldots, L_1)),$$

where $k(\bar{u})$ now denotes $k(\bar{u}_{d-1}, \ldots, \bar{u}_1)$. Let $M_k(X) \in A[X]$ for $k = 1, \ldots, t$ be forms of degree d_k such that the residues of $M_1(x_i/x), \ldots, M_t(x_i/x)$ modulo $\mathfrak{p}_s$ form a basis of $\mathfrak{R}^1/\mathfrak{p}_s$ over $k(\bar{u}) = B/mB$. If G is a form in $A[X]$ of degree $g \geqslant \max\{d_1, \ldots, d_t\}$, then

$$G(\theta_{si}) = \sum_{k=1,\ldots,t} \alpha_k(\Pi(\theta_{si}))^{g-d_k} M_k(\theta_{si})$$

for some $\alpha_1, \ldots, \alpha_t \in k(\bar{u})$, for $\Pi(\theta_{si}) = 1$. Letting

$$0 \longrightarrow K \longrightarrow k(\bar{u})[X_1, \ldots, X_n] \longrightarrow k(\bar{u})[\theta_{s1}, \ldots, \theta_{sn}] \longrightarrow 0$$

be the exact where $X_i \longrightarrow \theta_{si}$, $k(\bar{u})[X]_g/K_g$ is of dimension t over $k(\bar{u})$ for $g \geqslant \max\{d_1, \ldots, d_t\}$. $K \supset (p_s, L_{d-1}, \ldots, L_1)$ by the correspondence between $\mathfrak{p}_s$ and p_s. Let $G \in K_g$. There is a unit β in $k(\bar{u})$ such that $\beta G \in k[\bar{u}][X]_g$, and there are $F_j \in k[\bar{u}][X]$ for $j = 1, \ldots, d-1$ such that

$$E' = \Pi^c \beta G = \sum_{j=1,\ldots,d-1} (\Theta_j - \bar{u}_j \Pi) F_j \in k[X]_{g+c}$$

where c is the degree of $\bar{u}$ in βG. Let $E \in A[X]_{g+c}$ be a representative of E'. $E(x_i/x) \in \mathfrak{p}_s$, so $E' \in p_s$. Thus $\Pi^c G \in (p_s, L_{d-1}, \ldots, L_1)$. Inductively Π is contained in no minimal prime ideal of $(p_s, L_{d-1}, \ldots, L_j)$. For let P be such a minimal prime ideal and suppose $\Pi \in P$. Then $\Theta_j \in P$, and inductively by dimension, P is a minimal prime ideal of $(p_s, L_{d-1}, \ldots, L_{j+1}, \Pi)$ which is a contradiction to the choice of Θ_j. $(p_s, L_d, \ldots, L_1)$ being of dimension one, G is contained in every primary component of $(p_s, L_d, \ldots, L_1)$ except perhaps the primary component belonging to I, $K_g = (p_s, L_d, \ldots, L_1)_g$ for all large enough values of g, and by comparison of the Hilbert polynomials, $t = e_I(k[X]/p_s)$.

Apply the first part of the proof of Lemma 1 *to* $\mathfrak{R}^1$ *over* $B = B_1$, and obtain

$$\mu_A(f) = \mu_B(f) = \sum_{i=1,\ldots,r} e_I(k[X]/p_i)\mu_{\mathfrak{R}_i}(f).$$

4. The valuation formula. Let A be a local ring with maximal ideal m. For a definition of a valuation of A, finite on A and centered at a prime ideal of A, see [2, §1]. By the additivity formula $\mu_A(f) = \Sigma_p \lambda_p(f) e_m(A/p)$ where the sum ranges over all prime ideals p of A which are of depth equal to the dimension of A. Assume that A is nonimbedded. Then the prime ideals p are all of height one, but they do not necessarily include all the prime ideals of height one. Then also $\lambda_p(Af)$ is a finite sum of finite rank one discrete valuations centered at p.

As an example, let A be an entire factorial ring of dimension greater than one. Let $\{v_\iota\}_{\iota \in I}$ be the set of prime divisors of type one of A, and let p_ι be a prime element of A with $v_\iota(p_\iota) = 1$. Let w_1 and w_2 be two distinct prime divisors of A centered at m, let $a_\iota = w_1(p_\iota)$ and $b_\iota = w_2(p_\iota)$, and then $w_1 = \Sigma_\iota a_\iota v_\iota$ and $w_2 = \Sigma_\iota b_\iota v_\iota$. Let $c_\iota = \min\{a_\iota, b_\iota\}$. Then $\Sigma_\iota c_\iota v_\iota \geqslant w_1$, $\Sigma_\iota c_\iota v_\iota \neq w_1$, and $\Sigma_\iota c_\iota v_\iota$ is not a sum of valuations centered at m.

THEOREM. *Let A be a local ring with maximal ideal m. There are integral valued valuations $v_1, \ldots, v_s$ finite on A centered at m, and there are positive integers $n_1, \ldots, n_s$ such that for every regular element f of A,*

$$\mu_A(f) = n_1 v_1(f) + \cdots + n_s v_s(f).$$

If A is nonimbedded if $\mu_A(f) = n_1 v_1(f) + \cdots + n_s v_s(f)$ for all regular elements f of A, if the valuations $v_1, \ldots, v_s$ are independent, and if the ideal generated by each $v_i(A)$ is all of the integers, then the valuations $v_1, \ldots, v_s$ and the integers $n_1, \ldots, n_s$ are unique. (If A is of dimension zero, μ_A is the trivial valuation: $\mu_A(f) = \infty$ if $f \in m$ and $\mu_A(f) = 0$ if $f \notin m$.)

The *proof of the formula* is now straightforward. By Lemma 2, A can be assumed to be entire. It may also be assumed that the residue field of A is infinite. In fact let $A[x]$ be the polynomial ring in one variable over A, let $S = A[x] \sim mA[x]$, and let $A(x) = S^{-1}A[x]$, a local ring with maximal ideal $m \cdot A(x)$ and residue field $A(x)/mA(x) = k(x)$ a simple transcendental extension of $k = A/m$. Then $\mu_A = \mu_{A(x)}$, for $A(x)/A(x)f \simeq (A/Af)(x)$ and letting $B = A/Af$

$$G_{mB(x)}B(x) = \sum_{n \geqslant 0} \frac{m^n B(x)}{m^{n+1}B(x)} \simeq \sum_{n \geqslant 0} \frac{m^n}{m^{n+1}} \otimes_A B(x)$$
$$\simeq \sum_{n \geqslant 0} \frac{m^n + Af}{m^{n+1} + Af} \otimes_k k(x) \simeq (G_m B) \otimes_k k(x),$$

so the multiplicities of A/Af and of $A(x)/A(x)f$ are equal. A valuation of $A(x)$ restricted to A remains a valuation. By Lemma 4, A can be assumed to be one dimensional, by Lemma 3, A can be assumed to be normal, and apply the Corollary of Proposition 2 to obtain the formula.

The *proof of the unicity* uses a slight generalization of the approximation theorem. Define two valuations of A to be *equivalent* if there is an order isomorphism and the usual commutative diagram, and to be *independent* if they are not equivalent.

LEMMA. Let *Q be a noetherian nonimbedded ring which is its own total quotient ring. Let $v_1, \ldots, v_s$ be independent rank one valuations of Q, let $u_1, \ldots, u_s \in Q$ and let $\alpha_i \in v_i(A)$ be finite for $i = 1, \ldots, s$. There is an element u of Q such that $v_i(u - u_i) = \alpha_i$ for $i = 1, \ldots, s$.*

PROOF. $Q = Q_1 \oplus \cdots \oplus Q_n$ where Q_j is a local ring of dimension zero, and let

$$\mathfrak{N}_j = Q_1 \oplus \cdots \oplus Q_{j-1} \oplus \mathfrak{N}_j \oplus Q_{j+1} \oplus \cdots \oplus Q_n$$

where $\mathfrak{N}_j$ is the nil radical of Q_j. Let $v_1, \ldots, v_t$ be all of the valuations $v_1, \ldots, v_s$ which have $N_{v_i} = N_1$. Then $v_1, \ldots, v_t$ are naturally independent valuations of $Q/N_1 = k_1$. By the approximation theorem for a field [7, Theorem 18, p. 45], there is an element u_1' of Q_1 with $v_i(u_1' - \text{proj}_1\, u_i) = \alpha_i$ for $i = 1, \ldots, t$. Repeat this for each N_j, obtaining $u_j' \in Q_j$ for $2 \leqslant j \leqslant n$. Let $u = u_1' \oplus \cdots \oplus u_n'$, and the proof of lemma is complete.

A is assumed to be nonimbedded. Suppose $n_1v_1 + \cdots + n_sv_s \geqslant 0$ where $v_1, \ldots, v_s$ are independent nontrivial rank one valuations finite on A. It is to be seen that $n_1 \geqslant 0, \ldots, n_{s-1} \geqslant 0$ and $n_s \geqslant 0$. Let $u = f/g \in QA$ where f and g are elements of A, such that for some i, $v_i(u) > 0$ and $v_j(u) = 0$ for $j \neq i$. Then $v_i(f) > v_i(g)$, $v_j(f) = v_j(g)$ for $j \neq i$, $n_i(v_i(f) - v_i(g)) \geqslant 0$ and $n_i \geqslant 0$.

EXAMPLE. Let

$$A = \mathbf{C}[x, y, z]_{(x,y,z)} = \mathbf{C}[X, Y, Z]_{(X,Y,Z)}/(XY - Z^3)$$

which is normal, analytically irreducible and Cohen-Macaulay. By direct computation $\mu_A(x) = \mu_A(y) = 3$, $\mu_A(x + y) = 2$, and μ_A is not a valuation. In fact, $\mu_A = v_x + v_y$ where $\mathbf{C}(y/z)[z]_{(z)}$ and $\mathbf{C}(x/z)[z]_{(z)}$ are the valuation rings of v_x and v_y respectively. Note that neither x nor y are superficial elements of A.

EXAMPLE. Let

$$A = k[w, x, y, z]_{(w,x,y,z)} = k[s^4, s^3t, st^3, t^4]_{(s^4, s^3t, st^3, t^4)} \subset k[s, t],$$

the polynomial ring in two variables over a field k. $IA = k[s^4, s^3t, s^2t^2, st^3, t^4]$, $\mathcal{V}_A = \{(s^4, s^3t, st^3, t^4)\}$ and A is not Cohen-Macaulay. A is the localization of a projective (graded) ring, and by Proposition 2, §1, $\mu_A = e_m(A)v_A$ where v_A is the order valuation of A. By direct computation $\mu_A(x) = 4$, so $e_m(A) = 4$. Also $\mathfrak{R} = k(s/t)[t^4]_{(t^4)}$ which verifies the formula of the theorem for this example.

REFERENCES

1. J. Hornell, *Intersection theory in an equicharacteristic regular local ring and the relative intersection theory,* Proc. Amer. Math. Soc. **36** (1972), 8–12. MR 46#9040.

2. ———, *Divisorial complete intersections,* Pacific J. Math. **45** (1973), 217–227. MR 47 #6668.

3. M. Nagata, *Local rings,* Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR 27 #5790.

4. D. G. Northcott, *Lessons on rings, modules and multiplicities,* Cambridge, Univ. Press, London, 1968. MR 38 #144.

5. D. G. Northcott, *The neighborhoods of a local ring*, J. London Math. Soc. 30 (1955), 360–375. MR 17, 86.

6. J.-P. Serre, *Algèbre locale. Multiplicités*, Lecture Notes in Math., vol. 11, Springer-Verlag, Berlin and New York, 1965. MR 34 #1352.

7. O. Zariski and P. Samuel, *Commutative algebra.* Vol. II, University Ser. in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR 22 #11006.

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